Predicativity of the Mahlo Universe in Type Theory

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TYPES

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Higher universes and inductive-recursive definitions

- The super-universe (Palmgren)
- The Mahlo universe (Setzer)
- General inductive-recursive definitions (Dybjer, Setzer)

Are they constructive in the sense of Martin-Löf 1979? Are they predicative in Martin-Löf's extended sense?

- Palmgren's paradox: adding a natural elimination rule for the Mahlo universe yields an inconsistency.

Martin-Löf type theory 1986

Two levels:

Theory of types (LF) Dependent type theory with dependent function types $(x : \sigma) \to \tau$, a type Set, and for each $A : \mathrm{Set}$ a type A of elements.

Theory of sets Constants for standard set formers $\Pi, \Sigma, 0, 1, 2, \mathrm{N}, \mathrm{W}, \mathrm{Id}, \ldots$ and their introductory and eliminatory constants. Equations for the computation rules for eliminatory constants.

The theories **IR**, **IIRD** (Dybjer, Setzer 1999, etc) are based on **LF**. The theory $\mathbf{TT}^{\mathrm{M}}$ of this talk is also based on **LF**.

The external Mahlo universe Set

A super-universe is a universe closed under the next-universe operator

$$(-)^+ : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set})$$

Similarly, there are super-super-universes, etc.
A further generalization is to build universes $(\mathrm{U}\, f_0\, f_1, \mathrm{T}\, f_0\, f_1)$ closed under arbitrary family operators

$$f : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set})$$

This turns Set into a Mahlo universe with $(\mathrm{U}\, f_0\, f_1, \mathrm{T}\, f_0\, f_1)$ as subuniverses, where f is split into two components:

$$\begin{aligned} f_0 &: (X_0 : \mathrm{Set}) \to (X_0 \to \mathrm{Set}) \to \mathrm{Set} \\ f_1 &: (X_0 : \mathrm{Set}) \to (X_1 : X_0 \to \mathrm{Set}) \to f_0\, X_0\, X_1 \to \mathrm{Set} \end{aligned}$$

Subuniverses of $\mathcal{S}et$ in **LF**

Introduction rules for the codes $(\mathrm{c}_0, \mathrm{c}_1)$ for the family operator (f_0, f_1). We omit the arguments for the family operator *parameter*.

$$\begin{array}{lcl} \mathrm{c}_0 & : & (x_0 : \mathrm{U}\, f_0\, f_1) \to (\mathrm{T}\, f_0\, f_1\, x_0 \to \mathrm{U}\, f_0\, f_1) \\ & \to & \mathrm{U}\, f_0\, f_1 \\ \mathrm{c}_1 & : & (x_0 : \mathrm{U}\, f_0\, f_1) \to (x_1 : \mathrm{T}\, f_0\, f_1\, x_0 \to \mathrm{U}\, f_0\, f_1) \\ & \to & \mathrm{T}\, f_0\, f_1\, (c_0\, x_0\, x_1) \to \mathrm{U}\, f_0\, f_1 \end{array}$$

Equality rules:

$$\begin{array}{rcl} \mathrm{T}\, f_0\, f_1\, (\mathrm{c}_0\, x_0\, x_1) & = & f_0\, (\mathrm{T}\, f_0\, f_1\, x_0)\, ((\mathrm{T}\, f_0\, f_1) \circ x_1) \\ \mathrm{T}\, f_0\, f_1\, (\mathrm{c}_1\, x_0\, x_1\, t) & = & f_1\, (\mathrm{T}\, f_0\, f_1\, x_0)\, ((\mathrm{T}\, f_0\, f_1) \circ x_1)\, t \end{array}$$

We also have constructors for codes for the standard set formers. We call the resulting theory $\mathbf{TT}^{\mathrm{M}}$.

Mahlo is predicative, after all

We suggest an answer to this question by

- building a "predicative" (inductively generated) model of $\mathbf{TT}^{\mathsf{M}}$ in classical set theory (**ZFC**) extended with
  - a Mahlo cardinal M
  - and an inaccessible cardinal $\mathsf{I} > \mathsf{M}$
- providing meaning explanations for $\mathbf{TT}^{\mathsf{M}}$ extending and slightly modifying those in Martin-Löf 1979.

Inductive definitions via rule sets (Aczel 1977)

A *rule* on a base set U is a pair of sets $u \subseteq U$ and $v \in U$ written

$$\frac{u}{v}$$

Let Φ be a set of rules on U. A set $w \subseteq U$ is Φ*-closed* iff

$$\frac{u}{v} \in \Phi \text{ and } u \subseteq w \text{ implies } v \in w.$$

There is a least Φ-closed set

$$I(\Phi) = \bigcap \{w \subseteq U \mid w \ \Phi\text{−closed}\},$$

the set inductively defined by Φ. (An impredicative definition!)

Inductive definition of Tarski-style subuniverses $\mathcal{U}\, f_0\, f_1$

Let M be a Mahlo cardinal and

$$f : \mathcal{F}am(\mathrm{V}_{\mathrm{M}}) \to \mathcal{F}am(\mathrm{V}_{\mathrm{M}})$$

The Mahlo property implies that there is inaccessible $\kappa_f < \mathrm{M}$ such that f restricts to a function

$$\mathcal{F}am(\mathrm{V}_{\kappa_f}) \to Fam(\mathrm{V}_{\kappa_f})$$

The following rule set on $\mathrm{V}_{\kappa_f} \times \mathrm{V}_{\kappa_f}$ inductively generates the graph of the decoding function $\mathcal{T}\, f_0\, f_1$ with domain $\mathcal{U}\, f_0\, f_1$:

$$\{\frac{\{(x, X)\} \cup \{(y\, z, Y\, z) | z \in X\}}{(\mathrm{c}_0\, x\, y, f_0\, X\, Y)} \mid x, X \in \mathrm{V}_{\kappa_f}, y, Y : X \to \mathrm{V}_{\kappa_f}\}$$

$$\cup$$

$$\{\frac{\{(x, X)\} \cup \{(y\, z, Y\, z) | z \in X\}}{(\mathrm{c}_1\, x\, y\, t, f_1\, X\, Y\, t)} \mid x, X \in \mathrm{V}_{\kappa_f}, y, Y : X \to \mathrm{V}_{\kappa_f}, t \in f_0\, X\, Y\}$$

$$\cup$$

$$\vdots$$

Inductive definition of the Mahlo universe $\mathcal{S}et$

The following rule set on $\mathrm{V_M}$ inductively generates $\mathcal{S}et$:

$$\{\frac{\{f_0\,(\mathcal{T}\,f_0\,f_1\,x_0)\,((\mathcal{T}\,f_0\,f_1)\circ x_1)\mid (x_0,x_1)\in\mathcal{F}am(\mathcal{U}\,f_0\,f_1)\}\cup\{f_1\,(\mathcal{T}\,f_0\,f_1\,x_0)\,((\mathcal{T}\,f_0\,f_1)\circ x_1)\,t\mid (x_0,x_1)\in\mathcal{F}am(\mathcal{U}\,f_0\,f_1),t\in f_0\,(\mathcal{T}\,f_0\,f_1\,x_0)\,((\mathcal{T}\,f_0\,f_1)\circ x_1)\}}{\mathcal{U}\,f_0\,f_1}$$

$$\mid f:\mathcal{F}am(\mathrm{V_M})\to\mathcal{F}am(\mathrm{V_M})\}$$

$$\cup$$

$$\vdots$$

We add $\mathcal{U}\,f_0\,f_1$ to $\mathcal{S}et$ whenever we already know that f (family) composed with $\mathcal{T}\,f_0\,f_1$ yields a function

$$\mathcal{F}am(\mathcal{U}\,f_0\,f_1)\to\mathcal{F}am(\mathcal{S}et)$$

This yields a model of $\mathbf{TT}^{\mathrm{M}}$.

Meaning explanations for $\mathbf{TT}^{\mathrm{M}}$

We assume the canonical forms, computation rules, and matching conditions for the standard set formers (Martin-Löf 1979) adapted to the logical framework version (Martin-Löf 1986). We add:

- New canonical forms:

$$\mathrm{U}\, f_0\, f_1$$

$$\mathrm{c}_0\, a_0\, a_1\,,\, \mathrm{c}_1\, a_0\, a_1\, b$$

- New computation rules:

$$\begin{aligned} \mathrm{T}\, f_0\, f_1\, (\mathrm{c}_0\, x_0\, x_1) &= f_0\, (\mathrm{T}\, f_0\, f_1\, x_0)\, ((\mathrm{T}\, f_0\, f_1) \circ x_1) \\ \mathrm{T}\, f_0\, f_1\, (\mathrm{c}_1\, x_0\, x_1\, t) &= f_1\, (\mathrm{T}\, f_0\, f_1\, x_0)\, ((\mathrm{T}\, f_0\, f_1) \circ x_1)\, t \end{aligned}$$

Matching conditions for $\mathrm{U}\, f_0\, f_1 : \mathrm{Set}$

This judgment is valid under the conditions that

$$f_0\,(\mathrm{T}\, f_0\, f_1\, x_0)\,((\mathrm{T}\, f_0\, f_1) \circ x_1) : \mathrm{Set}$$

$$f_1\,(\mathrm{T}\, f_0\, f_1\, x_0)\,((\mathrm{T}\, f_0\, f_1) \circ x_1)\, t : \mathrm{Set}$$

in the context

$$x_0 : \mathrm{U}\, f_0\, f_1, x_1 : \mathrm{T}\, f_0\, f_1\, x_0 \to \mathrm{U}\, f_0\, f_1, t : f_0\,(\mathrm{T}\, f_0\, f_1\, x_0)\,((\mathrm{T}\, f_0\, f_1) \circ x_1)$$

Note the difference between this condition and the assumption of U-formation:

$$f : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set})$$

Well-foundedness

The repeated process of lazily computing canonical forms and checking matching conditions must be well-founded. For example

- $c : \mathrm{N}$ is only valid if the process of computing successive canonical forms of c produces finitely many successors and ends with a final matching $0 : \mathrm{N}$. (If we get an infinite sequence of successors, then the judgment is not valid.)
- $c : \mathrm{W}\, A\, B$ must generate a well-founded tree of matchings of canonical forms. The root of the tree is the matching of $\sup a\, b : \mathrm{W}\, A\, B$ and the subtrees are matchings of the canonical forms of $a : A$ and of $b\, x : \mathrm{W}\, A\, B$ for each $x : B\, a$.

Well-foundedness is a non-trivial issue for the Mahlo universe. Cf Palmgren's paradox.

Justification of the rules

- *Meaning explanations* express what the judgments of type theory *mean* (Martin-Löf 1979).
- *Justification of the rules* is a *second step*. It's too much to ask for absolute guarantees for the validity of the inference rules. But we can still provide evidence why we believe they are correct. Martin-Löf 1979:

> *But there are also certain limits to what verbal explanations can do when it comes to justifying axioms and rules of inference. In the end, everybody must understand for himself.*

- We may use any means at our disposal, e g mathematical model building in set theory. When we justify the rules of type theory with Set as a Mahlo universe it parallels the proof that the set-theoretic model is a model of $\mathbf{TT}^{\mathrm{M}}$.